Some identities of symmetry for the generalized Bernoulli numbers and polynomials

By

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Abstract. In this paper, by the properties of p-adic invariant integral on $\mathbb{Z}_p$, we establish various identities concerning the generalized Bernoulli numbers and polynomials. From the symmetric properties of p-adic invariant integral on $\mathbb{Z}_p$, we give some interesting relationship between the power sums and the generalized Bernoulli polynomials.

2000 Mathematics Subject Classification: 11B68, 11M38, 11S80.
Key Words and Phrases: p-adic invariant integral, Bernoulli numbers, Bernoulli polynomials.

§1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [6])}. \tag{1}$$

From the definition (1), we have

$$I_1(f_1) = I_1(f) + f'(0), \text{ where } f'(0) = \tfrac{df(x)}{dx}|_{x=0} \text{ and } f_1(x) = f(x+1). \tag{2}$$

Let $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$. Then we can derive the following equation (3) from (2).

$$I(f_n) = I(f) + \sum_{i=0}^{n} f'(i), \quad \text{(see [6])}. \tag{3}$$

It is well known that the ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{ see [1-25] }),$$

and the Bernoulli number B_n are defined as $B_n=B_n(0)$.

Let d a fixed positive integer. For $n\in\mathbb{N}$, we set

$$X=X_d=\varprojlim_{N}\left(\mathbb{Z}/dp^N\mathbb{Z}\right),\quad X_1=\mathbb{Z}_p;$$

$$X^*=\bigcup_{\substack{0<a<dp,\\(a,p)=1}}(a+dp\mathbb{Z}_p);$$

$$a+dp^N\mathbb{Z}_p=\{\ x\in X|\ x\equiv a\ (\mathrm{mod}\ dp^N)\ \},$$

where $a\in\mathbb{Z}$ lies in $0\le a<dp^N$. In [6], it is known that

$$\int_X f(x)dx=\int_{\mathbb{Z}_p}f(x)dx,\quad \text{for } f\in UD(\mathbb{Z}_p).$$

Let us take $f(x)=e^{tx}$. Then we have

$$\int_{\mathbb{Z}_p}e^{tx}dx=\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

Thus, we note that

$$\int_{\mathbb{Z}_p}x^n dx=B_n,\quad n\in\mathbb{Z}_+,\quad (\text{see [1-25]}).$$

Let χ be the Dirichlet's character with conductor $d\in\mathbb{N}$. Then the generalized Bernoulli polynomials attached to χ are defined as

$$\sum_{a=1}^{d}\frac{\chi(a)te^{at}}{e^{dt}-1}e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^n}{n!},\quad (\text{ see [22] }),\tag{4}$$

and the generalized Bernoulli numbers attached to χ, $B_{n,\chi}$ are defined as $B_{n,\chi}=B_{n,\chi}(0)$.

In this paper, we investigate the interesting identities of symmetry for the generalized Bernoulli numbers and polynomials attached to χ by using the properties of p-adic invariant integral on $\mathbb{Z}_p$. Finally, we will give relationship between the power sum polynomials and the generalized Bernoulli numbers attached to χ.

§2. **Symmetry of power sum and the generalized Bernoulli polynomials**

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$. From (3), we note that

$$\int_X \chi(x)e^{xt}dx = \frac{t\sum_{i=0}^{d-1}\chi(i)e^{it}}{e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}, \tag{5}$$

where $B_{n,\chi}(x)$ are n-th generalized Bernoulli numbers attached to χ. Now, we also see that the generalized Bernoulli polynomials attached to χ are given by

$$\int_X \chi(y)e^{(x+y)t}dy = \frac{\sum_{i=0}^{d-1}\chi(i)e^{it}}{e^{dt}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!}. \tag{6}$$

By (5) and (6), we easily see that

$$\int_X \chi(x)x^n dx = B_{n,\chi}, \quad \text{and} \quad \int_X \chi(y)(x+y)^n dy = B_{n,\chi}(x). \tag{7}$$

From (6), we have

$$B_{n,\chi}(x) = \sum_{\ell=0}^{n}\binom{n}{\ell}B_{\ell,\chi}x^{n-\ell}. \tag{8}$$

From (6), we can also derive

$$\int_X \chi(x)e^{xt}dx = \sum_{i=0}^{d-1}\chi(i)\frac{t}{e^{dt}-1}e^{(\frac{i}{d})dt} = \sum_{n=0}^{\infty}\Big(d^n\sum_{i=0}^{d-1}\chi(i)B_n(\frac{i}{d})\Big)\frac{t^n}{n!}.$$

Therefore, we obtain the following lemma.

Lemma1. *For $n \in \mathbb{Z}_+$, we have*

$$\int_X \chi(x)x^n dx = B_{n,\chi} = d^n\sum_{i=0}^{d-1}\chi(i)B_i\Big(\frac{i}{d}\Big).$$

We observe that

$$\frac{1}{t}\Big(\int_X \chi(x)e^{(nd+x)t}dx - \int_X e^{xt}\chi(x)dx\Big) = \frac{nd\int_X \chi(x)e^{xt}dx}{\int_X e^{ndxt}dx} = \frac{e^{ndt}-1}{e^{dt}-1}\Big(\sum_{i=0}^{d-1}\chi(i)e^{it}\Big). \tag{9}$$

Thus, we have

$$\frac{1}{t}\Big(\int_X \chi(x)e^{(nd+x)t}dx - \int_X e^{xt}dx\Big) = \sum_{k=0}^{\infty}\Big(\sum_{\ell=0}^{nd-1}\chi(\ell)\ell^k\Big)\frac{t^k}{k!}. \tag{10}$$

Let us define the p-adic functional $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{\ell=0}^{n} \chi(\ell)\ell^k, \quad \text{for} \quad k \in \mathbb{Z}_+. \tag{11}$$

By (10) and (11), we see that

$$\frac{1}{t}\Big(\int_X \chi(x)e^{(nd+x)t}dx - \int_X e^{xt}dx\Big) = \sum_{n=0}^{\infty}\Big(T_k(\chi, nd-1)\Big)\frac{t^k}{k!}. \tag{12}$$

By using Taylor expansion in (12), we have

$$\int_X \chi(x)(dn+x)^k dx - \int_X \chi(x)x^k dx = kT_{k-1}(\chi, nd-1), \quad \text{for} \quad k, n, d \in \mathbb{N}. \tag{13}$$

That is,

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd-1).$$

Let $w_1, w_2, d \in \mathbb{N}$. Then we consider the following integral equation

$$\begin{aligned} &\frac{d\int_X\int_X \chi(x_1)\chi(x_2)e^{(w_1x_1+w_2x_2)t}dx_1dx_2}{\int_X e^{dw_1w_2xt}dx} \\ &= \frac{t(e^{dw_1w_2t}-1)}{(e^{w_1dt}-1)(e^{w_2dt}-1)}\Big(\sum_{a=0}^{d-1}\chi(a)e^{w_1at}\Big)\Big(\sum_{b=0}^{d-1}\chi(b)e^{w_2bt}\Big). \end{aligned} \tag{14}$$

From (9) and (12), we note that

$$\frac{dw_1\int_X \chi(x)e^{xt}dx}{\int_X e^{dw_1xt}dx} = \sum_{k=0}^{\infty}\Big(T_k(\chi, dw_1-1)\Big)\frac{t^k}{k!}. \tag{15}$$

Let us consider the p-adic functional $T_\chi(w_1, w_2)$ as follows:

$$T_\chi(w_1, w_2) = \frac{d\int_X\int_X \chi(x_1)\chi(x_2)e^{(w_1x_1+w_2x_2+w_1w_2x)t}dx_1dx_2}{\int_X e^{dw_1w_2x_3t}dx_3}. \tag{16}$$

Then we see that $T_\chi(w_1, w_2)$ is symmetric in w_1 and w_2, and

$$T_\chi(w_1, w_2) = \frac{t(e^{dw_1w_2t}-1)e^{w_1w_2xt}}{(e^{w_1dt}-1)(e^{w_2dt}-1)}\Big(\sum_{a=0}^{d-1}\chi(a)e^{w_1at}\Big)\Big(\sum_{b=0}^{d-1}\chi(b)e^{w_2bt}\Big). \tag{17}$$

By (16) and (17), we have

$$\begin{aligned}
T_\chi(w_1,w_2) &= \Big(\frac{1}{w_1}\int_X \chi(x_1)e^{w_1(x_1+w_2x)t}dx_1\Big)\Big(\frac{dw_1\int_X \chi(x_2)e^{w_2x_2t}dx_2}{\int_X e^{dw_1w_2xt}dx}\Big) \\
&= \Big(\frac{1}{w_1}\sum_{i=0}^{\infty} B_{i,\chi}(w_2x)\frac{w_1^i t^i}{i!}\Big)\Big(\sum_{k=0}^{\infty} T_k(\chi, dw_1-1)\frac{w_2^k t^k}{k!}\Big) \\
&= \frac{1}{w_1}\Big(\sum_{\ell=0}^{\infty}(\sum_{i=0}^{\ell}\frac{B_{i,\chi}(w_2x)T_{\ell-i}(\chi,dw_1-1)w_1^i w_2^{\ell-i}\ell!}{i!(\ell-i)!})\frac{t^\ell}{\ell!}\Big) \\
&= \sum_{\ell=0}^{\infty}\Big(\sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}(w_2x)T_{\ell-i}(\chi,dw_1-1)w_1^{i-1}w_2^{\ell-i}\Big)\frac{t^\ell}{\ell!}.
\end{aligned} \tag{18}$$

From the symmetric property of $T_\chi(w_1,w_2)$ in w_1 and w_2, we note that

$$\begin{aligned}
T_\chi(w_1,w_2) &= \Big(\frac{1}{w_2}\int_X \chi(x_2)e^{w_2(x_2+w_1x)t}dx_2\Big)\Big(\frac{dw_2\int_X \chi(x_1)e^{w_1x_1t}dx_1}{\int_X e^{dw_1w_2xt}dx}\Big) \\
&= \Big(\frac{1}{w_2}\sum_{i=0}^{\infty} B_{i,\chi}(w_1x)\frac{w_2^i t^i}{i!}\Big)\Big(\sum_{k=0}^{\infty} T_k(\chi, dw_2-1)\frac{w_1^k t^k}{k!}\Big) \\
&= \frac{1}{w_2}\Big(\sum_{\ell=0}^{\infty}(\sum_{i=0}^{\ell}\frac{B_{i,\chi}(w_1x)w_2^iT_{\ell-i}(\chi,dw_2-1)w_1^{\ell-i}\ell!}{i!(\ell-i)!})\frac{t^\ell}{\ell!}\Big) \\
&= \sum_{\ell=0}^{\infty}\Big(\sum_{i=0}^{\ell}\binom{\ell}{i}w_2^{i-1}w_1^{\ell-i}B_{i,\chi}(w_1x)T_{\ell-i}(\chi,dw_2-1)\Big)\frac{t^\ell}{\ell!}.
\end{aligned} \tag{19}$$

By comparing the coefficients on the both sides of (18) and (19), we obtain the following theorem.

Theorem 2. *For $w_1, w_2, d \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}(w_2x)T_{\ell-i}(\chi,dw_1-1)w_1^{i-1}w_2^{\ell-i} \\
&= \sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}(w_1x)T_{\ell-i}(\chi,dw_2-1)w_2^{i-1}w_1^{\ell-i}.
\end{aligned}$$

Let $x=0$ in Theorem 2. Then we have

$$\begin{aligned}
&\sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}T_{\ell-i}(\chi,dw_1-1)w_1^{i-1}w_2^{\ell-i} \\
&= \sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}T_{\ell-i}(\chi,dw_2-1)w_2^{i-1}w_1^{\ell-i}.
\end{aligned}$$

By (15) and (17), we also see that

$$\begin{aligned}
T_\chi(w_1,w_2) &= \Big(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\Big)\Big(\frac{dw_1\int_X \chi(x_2)e^{w_2x_2t}dx_2}{\int_X e^{dw_1w_2xt}dx}\Big)\\
&= \Big(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\Big)\Big(\frac{e^{dw_1w_2t}-1}{e^{w_2dt}-1}\Big)\Big(\sum_{i=0}^{d-1}\chi(i)e^{w_2it}\Big)\\
&= \Big(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\Big)\Big(\sum_{\ell=0}^{w_1-1}\sum_{i=0}^{d-1}e^{w_2(i+\ell d)t}\chi(i+\ell d)\Big)\\
&= \Big(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\Big)\Big(\sum_{i=0}^{dw_1-1}e^{w_2it}\chi(i)\Big)\\
&= \frac{1}{w_1}\sum_{i=0}^{dw_1-1}\chi(i)\int_X \chi(x_1)e^{w_1(x_1+w_2x+\frac{w_2}{w_1}i)t}dx_1\\
&= \frac{1}{w_1}\sum_{i=0}^{dw_1-1}\chi(i)\sum_{k=0}^{\infty}B_{k,\chi}(w_2x+\frac{w_2}{w_1}i)\frac{w_1^kt^k}{k!}\\
&= \sum_{k=0}^{\infty}\Big(\sum_{i=0}^{dw_1-1}\chi(i)B_{k,\chi}(w_2x+\frac{w_2}{w_1}i)w_1^{k-1}\Big)\frac{t^k}{k!}.
\end{aligned}\tag{20}$$

From the symmetric property of $T_\chi(w_1,w_2)$ in w_1 and w_2, we can also derive the following equation.

$$\begin{aligned}
T_\chi(w_1,w_2) &= \Big(\frac{e^{w_1w_2xt}}{w_2}\int_X \chi(x_2)e^{w_2x_2t}dx_2\Big)\Big(\frac{dw_2\int_X \chi(x_1)e^{w_1x_1t}dx_1}{\int_X e^{dw_1w_2xt}dx}\Big)\\
&= \Big(\frac{e^{w_1w_2xt}}{w_2}\int_X \chi(x_2)e^{w_2x_2t}dx_2\Big)\Big(\frac{e^{dw_1w_2t}-1}{e^{w_1dt}-1}\Big)\Big(\sum_{i=0}^{d-1}\chi(i)e^{w_1it}\Big)\\
&= \Big(\frac{e^{w_1w_2xt}}{w_2}\int_X \chi(x_2)e^{w_2x_2t}dx_2\Big)\Big(\sum_{\ell=0}^{w_2-1}e^{w_1d\ell t}\Big)\Big(\sum_{i=0}^{d-1}\chi(i)e^{w_1it}\Big)\\
&= \frac{1}{w_2}\sum_{i=0}^{dw_2-1}\chi(i)\int_X \chi(x_2)e^{w_2(x_2+w_1x+\frac{w_1}{w_2}i)t}dx_2\\
&= \frac{1}{w_2}\sum_{i=0}^{dw_2-1}\chi(i)\sum_{k=0}^{\infty}B_{k,\chi}(w_1x+\frac{w_1}{w_2}i)\frac{w_2^kt^k}{k!}\\
&= \sum_{k=0}^{\infty}\Big\{\sum_{i=0}^{dw_2-1}\chi(i)B_{k,\chi}(w_1x+\frac{w_1}{w_2}i)w_2^{k-1}\Big\}\frac{t^k}{k!}.
\end{aligned}\tag{21}$$

By comparing the coefficients on the both sides of (20) and (21), we obtain the following theorem.

Theorem 3. *For $w_1, w_2, d \in \mathbb{N}$, we have*

$$\sum_{i=0}^{dw_1-1} \chi(i) B_{k,\chi}(w_2 x + \frac{w_2}{w_1} i) w_1^{k-1} = \sum_{i=0}^{dw_2-1} \chi(i) B_{k,\chi}(w_1 x + \frac{w_1}{w_2} i) w_2^{k-1}.$$

Remark. *Let $x = 0$ in* Theorem 3. *Then we see that*

$$\sum_{i=0}^{dw_1-1} \chi(i) B_{k,\chi}(\frac{w_2}{w_1} i) w_1^{k-1} = \sum_{i=0}^{dw_2-1} \chi(i) B_{k,\chi}(\frac{w_1}{w_2} i) w_2^{k-1}.$$

If we take $w_2 = 1$, then we have

$$\sum_{i=0}^{dw_1-1} \chi(i) B_{k,\chi}(\frac{i}{w_1}) w_1^{k-1} = \sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(w_1 i).$$

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